Mathematical Models and Methods in Applied Sciences

ADVECTION AND DISPERSION IN TIME AND SPACE

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Previous work showed how moving particles that rest along their trajectory lead to time-nonlocal advection-dispersion equations. If the waiting times have infinite mean, the model equation contains a fractional time derivative of order between 0 and 1. In this article, we propose a new advection-dispersion equation utilizing a fractional time derivative of order between 1 and 2. Solutions to this equation are obtained by subordination. The form of the time derivative is related to the probability distribution of particle waiting times. In principle, the distribution of random time that particles spend in an immobile state is easily measured, and then the order of the fractional time derivative can be inferred.

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1. Introduction

Continuous time random walks (CTRW) can be used to derive governing equations for anomalous diffusion.[11,12,13] The CTRW is a stochastic process model for the

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movement of an individual particle. In the long-time limit, the process converges to a simpler form whose probability densities solve the governing equation, leading to a useful model for anomalous diffusion. For a simple random walk with mean zero, finite variance particle jumps, the limit process is a Brownian motion $A(t)$ governed by the classical diffusion equation $\partial p/\partial t = \partial^2 p/\partial x^2$ where $p(x,t)$ is the probability density of the random variable $A(t)$. For symmetric infinite variance jumps (whose probability distribution is assumed regularly varying[10] with some index $0 < \alpha \leq 2$), the limit process $A(t)$ is an α-stable Lévy motion, and the governing equation becomes $\partial p/\partial t = \partial^\alpha p/\partial|x|^\alpha$.[8] When waiting times are introduced, the limiting process is altered via subordination. For infinite mean waiting times (whose probability distribution is assumed regularly varying with some index $0 < \gamma < 1$) the limit process is $A(E(t))$ where $E(t)$ is the inverse or hitting time process for the γ-stable subordinator. The process $E(t)$ counts the number of particle jumps by time $t \geq 0$, accounting for the waiting time between particle jumps. The governing equation becomes $\partial^\gamma p/\partial t^\gamma = \partial^\alpha p/\partial|x|^\alpha + \delta(x)t^{-\gamma}/\Gamma(1-\gamma)$.[11,12] Some applications[7] seem to require a time derivative of order $1 < \gamma \leq 2$. In this paper, we develop one such equation by extending the CTRW approach.

2. The model

In the usual CTRW formalism, the long-time limit for the waiting time process is a γ-stable subordinator $D(t)$.[11] Then the inverse Lévy process $E(t) = \inf\{x : D(x) > t\}$ counts the number of particle jumps by time $t \geq 0$, reflecting the fact that the time T_n of the nth particle jump and the number $N_t = \max\{n : T_n \leq t\}$ of jumps by time t are also inverse processes. When the waiting times between particle jumps have heavy tails with $0 < \gamma < 1$, subordination of the particle location process $A(t)$ via the inverse Lévy process $E(t)$ is necessary in the long-time limit to account for this, which leads to a time derivative of order γ in the governing equation.[12] When waiting times have heavy tails of order $1 < \gamma \leq 2$, meaning that the probability of waiting longer than t falls off like $t^{-\gamma}$, a different model is needed.[5] In this case, convergence of the waiting time process requires centering to the mean waiting time w, which is not necessary when $0 < \gamma < 1$. Accounting for this leads to a waiting time process $W(t) = D(t) + wt$ where $D(t)$ is a completely positively skewed[a] stable Lévy process with index γ, so that $W(t)$ is a Lévy process with drift. The drift ensures that $W(t) \to \infty$ with probability one as $t \to \infty$. Since $\gamma > 1$, the process $W(t)$ is not strictly increasing, so we use $\mathrm{Max}(t) = \sup\{W(u) : 0 \leq u \leq t\}$ to represent the particle jump times. Then the inverse or hitting time process $H(t) = \inf\{x : \mathrm{Max}(x) \geq t\}$ counts the number of particle jumps.

[a]The skew is irrelevant in the normal case $\gamma = 2$.

3. Hitting time density

The waiting time process $W(t)$ has characteristic function

$$E[e^{i\xi W(t)}] = e^{t(iw\xi + a(-i\xi)^\gamma)},$$

where $w > 0$ is the mean waiting time and a a shape variable akin to the variance. For the remainder of this paper we make the assumption $w = 1$, which entails no loss of generality, since this can always be achieved by a simple rescaling in time. In order to compute the density of the process H we use the fact that

$$P\{H(T) > s\} = P\{\mathrm{Max}(s) < T\}$$

for all $s, T \geq 0$. Theorem 1 in Baxter and Donsker (1957)[4] shows that the distribution of the maximum process $M(s,T) = P\{\mathrm{Max}(s) < T\}$ satisfies

$$\begin{aligned} u \int_0^\infty \int_0^\infty & e^{-us-\lambda T} d_T(M(s,T)) ds \\ &= \exp\left\{ \frac{1}{2\pi} \int_u^\infty \int_{-\infty}^\infty \frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))} \, d\xi \, dx \right\} \end{aligned} \quad (3.1)$$

for all $\lambda, u > 0$. The integrand has two poles in the upper complex halfplane, at $\xi = i\lambda$ and whenever $x = i\xi + a(-i\xi)^\gamma$. That the second equality holds only once, follows from the following Lemma investigating the inverse function of $az^\gamma - z$. This involves the following region. Let $a > 0$, $0 < \alpha \leq \pi/\gamma$ and

$$\Omega(\alpha) := \left\{ re^{i\theta} : -\alpha < \theta < \alpha \text{ and } \frac{\sin(\theta)}{a\sin(\gamma\theta)} < r^{\gamma-1} < \frac{\sin(\alpha)}{a\sin(\gamma\alpha)} \right\}$$

where we take $\sin(\alpha)/a\sin(\gamma\alpha) = \infty$ when $\alpha = \pi/\gamma$.

Lemma 3.1. *Let $a > 0, 1 < \gamma \leq 2$. There exists a unique holomorphic function $q : \mathbb{C} \setminus (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}] \to \Omega(\pi/\gamma)$ such that*

$$aq(z)^\gamma - q(z) = z.$$

Furthermore, there exists an analytic function F with $\sup_{t>0} t^{1-1/\gamma} e^{\zeta t} |F(t)| < \infty$ for all $0 < \zeta < a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$ such that $\int_0^\infty e^{-zt} F(t)\, dt = 1/q(z)$ for $z > 0$.

Proof. First we show uniqueness. Let $0 < \alpha < \pi/\gamma$,

$$\Gamma_\pm = \left\{ \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)} \right)^{\frac{1}{\gamma-1}} e^{\pm i\theta} : 0 \leq \theta \leq \alpha \right\}$$

and

$$\Gamma_r = \left\{ \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)} \right)^{\frac{1}{\gamma-1}} e^{i\theta} : -\alpha \leq \theta \leq \alpha \right\}.$$

Then $\partial\,\Omega(\alpha) = \Gamma := \Gamma_- + \Gamma_r - \Gamma_+$ is clearly a simply closed path around $\Omega(\alpha)$ (see for example Rudin,[16] Thm. 10.40). Let

$$p(z) = az^\gamma - z.$$

Then

$$\begin{aligned}
p\left(\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} e^{\pm i\theta}\right) &= a\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{\gamma}{\gamma-1}} e^{\pm i\gamma\theta} - \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} e^{\pm i\theta} \\
&= a\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{\gamma}{\gamma-1}} \cos(\gamma\theta) - \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} \cos(\theta) \\
&\quad \pm i\left(a\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{\gamma}{\gamma-1}} \sin(\gamma\theta) - \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} \sin(\theta)\right) \\
&= \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\theta)}{\sin(\gamma\theta)}\cos(\gamma\theta) - \cos(\theta)\right) \\
&= -\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\theta)}{\sin(\gamma\theta)}.
\end{aligned}$$

A quick calculation shows that for $0 < \theta < \pi/\gamma$, $\theta \mapsto \frac{\sin(\theta)}{\sin(\gamma\theta)}$ is an increasing function which implies that $\theta \mapsto \left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\theta)}{\sin(\frac{\gamma}{\gamma-1}(\gamma-1)\theta)}$ is also increasing. Since

$$\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{a\sin(\gamma\theta)}\right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\theta)}{\sin(\gamma\theta)} = a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$$

we have that the image of $\Gamma_\pm$ under p is a path on the negative real axis,

$$p(\Gamma_\pm) = \left[-\left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} \frac{\sin((\gamma-1)\alpha)}{\sin(\gamma\alpha)}, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}\right].$$

Investigating $p(\Gamma_r)$ we see that for $z \in \Gamma_r$,

$$\begin{aligned}
p(z) &= a\left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{\gamma}{\gamma-1}} e^{i\gamma\theta} - \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} e^{i\theta} \\
&= \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\alpha)}{\sin(\gamma\alpha)}(\cos(\gamma\theta) + i\sin(\gamma\theta)) - \cos(\theta) - i\sin(\theta)\right) \\
&= \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\alpha)\cos(\gamma\theta)}{\sin(\gamma\alpha)} - \cos(\theta) + i\left(\frac{\sin(\alpha)\sin(\gamma\theta)}{\sin(\gamma\alpha)} - \sin(\theta)\right)\right).
\end{aligned}$$

Using again that $\theta \mapsto \sin(\theta)/\sin(\gamma\theta)$ is increasing for $\theta > 0$, the imaginary part is positive iff θ is positive. Furthermore,

$$|p(z)| \geq \left(\frac{\sin(\alpha)}{a\sin(\gamma\alpha)}\right)^{\frac{1}{\gamma-1}} \left(\frac{\sin(\alpha)}{\sin(\gamma\alpha)} - 1\right)$$

for all $z \in \Gamma_r$, and this lower bound tends to infinity as $\alpha \to \pi/\gamma$. Hence we obtain for α large enough that $p(\Gamma_r)$ is a closed contour going once counterclockwise around the origin, which implies that $p(\Gamma)$ is a closed contour going once counterclockwise around the origin.

Fix $w \in \mathbb{C} \setminus (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}]$, and then choose $\alpha < \pi/\gamma$ (α depends on w) such that $p(\Gamma)$ goes around w. Using the counting formula for zeros and poles, we see that there is exactly one $z \in \Omega(\alpha)$ (see, for example, Remmert,[15] Theorem 13.2.2) such that $p(z) = w$. Furthermore (see for example Ahlfors,[1] p.153),

$$q(w) := p^{-1}(w) = \frac{1}{2\pi i}\int_\Gamma \zeta \frac{p'(\zeta)}{p(\zeta)-w}\,d\zeta$$

is a holomorphic function on $\mathbb{C} \setminus (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}]$.

Next, we show that $1/q$ is the Laplace transform of an analytic function f on $(0,\infty)$ with $t^{1-1/\gamma}e^{-\omega t}f(t)$ bounded for $\omega > -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$.

Since $aq(z)^\gamma - q(z) = z$ we have that for $|z|$ large enough, $(a+1)|q(z)|^\gamma > |aq(z)^\gamma - q(z)| = |z|$. Furthermore, $q(z) \in \Omega(\pi/\gamma)$ and thus $q(z)$ is bounded away from zero. Hence there exists $M > 0$ such that

$$M|q(z)|^\gamma \geq |z| \tag{3.3}$$

for all $z \notin (-\infty, -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}]$. Let

$$r(z) := \frac{d}{dz}\frac{1}{q(z)} = -q(z)^{-2}\frac{1}{a\gamma q(z)^{\gamma-1}-1}$$

using the fact that $dq/dz = 1/(dp/dz)$. The function $z \mapsto \frac{1}{a\gamma q(z)^{\gamma-1}-1}$ has a single pole at $z = -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$ and is bounded off a neighborhood of that pole. Choose $\omega < 0$ such that $\omega > -a(\gamma-1)(a\gamma)^{\frac{\gamma}{1-\gamma}}$. Then for $\Sigma_\alpha := \{re^{i\delta} : r > 0, -\alpha < \delta < \alpha\}$ we have that

$$\sup_{z\in\omega+\Sigma_{\delta+\pi/2}} |(z-\omega)r(z)| = \sup_{z\in\omega+\Sigma_{\delta+\pi/2}} \left|\frac{z-\omega}{q(z)^2}\frac{1}{a\gamma q(z)^{\gamma-1}-1}\right| < \infty$$

for all $0 < \delta < \pi/2$. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}$. By the analytic representation theorem for Laplace transforms (Arendt et al.[2] Theorem 2.6.1) there exists a holomorphic function $f : \mathbb{C}_+ \to \mathbb{C}$ such that $\sup_{z\in\mathbb{C}_+} |e^{-\omega z}f(z)| < \infty$ and

$$r(z) = \int_0^\infty e^{-zt}f(t)\,dt.$$

Furthermore,

$$\sup_{\Re(z)>0} |zz^{\frac{1}{\gamma}}r(z)| = \left|\frac{z^{2/\gamma}}{q(z)^2}\frac{z^{\frac{\gamma-1}{\gamma}}}{a\gamma q(z)^{\gamma-1}-1}\right| < \infty.$$

Using the complex representation theorem (Arendt et al.[2] Theorem 2.5.1 with $b = 1/\gamma$ and $q(z) = z^{\frac{1}{\gamma}}r(z)$) there exists $g \in C[0,\infty)$ with $\sup_{t>0} t^{-1/\gamma}|g(t)| < \infty$ such that

$$z^{1/\gamma}r(z) = z^{\frac{1}{\gamma}}\int_0^\infty e^{-zt}g(t)\,dt.$$

By the uniqueness of the Laplace transform $f = g$ and hence

$$\sup_{t\geq 0} |t^{-\frac{1}{\gamma}}e^{-\omega t}f(t)| < \infty.$$

Clearly, $F(t) = -f(t)/t \in L^1(0,\infty)$ and hence, using Fubini,

$$\int_0^\infty e^{-zt} F(t)\,dt = -\int_z^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds = -\int_z^\infty r(s)\,ds = 1/q(z)$$

for $z > 0$. Thus the Laplace transform of $F(t)$ is $1/q$ and the growth conditions follow with $\zeta = -\omega$. □

We are now in a position to simplify (3.1). Clearly, $z > 0$ implies $q(z) > 0$. Thus the second pole of the integrand of (3.1),

$$\frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))},$$

is at $\xi = iq(x)$. For $\Gamma_n = \{ne^{i\theta} : 0 \le \theta \le \pi\}$ there exists $M > 0$ such that

$$\begin{aligned} \lim_{n\to\infty} \quad & \frac{1}{2\pi} \int_{\Gamma_n} \left| \frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))} \right| d\xi \\ & \le \lim_{n\to\infty} \frac{1}{2\pi} \int_{\Gamma_n} \frac{M}{n^2}\,d\xi = \lim_{n\to\infty} M/2n = 0. \end{aligned}$$

Thus, we can apply the residue theorem and obtain

$$\begin{aligned} \frac{1}{2\pi} \quad & \int_{-\infty}^\infty \frac{\lambda}{\xi(\xi - i\lambda)} \frac{i\xi + a(-i\xi)^\gamma}{x(x - (i\xi + a(-i\xi)^\gamma))}\,d\xi \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{[-n,n]\cup\Gamma_n} \frac{\lambda}{(\xi - i\lambda)} \frac{-1 + a(-i\xi)^{\gamma-1}}{x(x - (i\xi + a(-i\xi)^\gamma))}\,d\xi \\ &= \lambda \frac{-1 + a\lambda^{\gamma-1}}{x(x - (-\lambda + a\lambda^\gamma))} + \frac{\lambda}{iq(x) - i\lambda} \frac{-1 + a(q(x))^{\gamma-1}}{x(-i + i\gamma a q(x)^{\gamma-1})} \\ &= \left(\frac{-\lambda + a\lambda^\gamma}{x(x + \lambda - a\lambda^\gamma)} + \frac{\lambda}{x(q(x) - \lambda)} \frac{-1 + a(q(x))^{\gamma-1}}{(1 - \gamma a q(x)^{\gamma-1})} \right). \end{aligned}$$

Since q is an inverse function we can compute its derivative

$$\frac{d}{dx} q(x) = \frac{1}{a\gamma q(x)^{\gamma-1} - 1}$$

as in the proof of Lemma 3.1. Thus for $u > a\lambda^\gamma - \lambda$,

$$\begin{aligned} \int_u^\infty \quad & \left(\frac{-\lambda + a\lambda^\gamma}{x(x + \lambda - a\lambda^\gamma)} \right) + \frac{\lambda}{x(q(x) - \lambda)} \frac{-1 + a(q(x))^{\gamma-1}}{(1 - \gamma a q(x)^{\gamma-1})} dx \\ &= [\ln(1 + (\lambda - a\lambda^\gamma)/x)]_u^\infty - \lambda \int_{q(u)}^\infty \frac{-1 + ax^{\gamma-1}}{(ax^\gamma - x)(x - \lambda)} dx \\ &= -\ln(1 + (\lambda - a\lambda^\gamma)/u) - [\ln(1 - \lambda/x)]_{q(u)}^\infty \\ &= \ln\left(\frac{1 - \lambda/q(u)}{1 + (\lambda - a\lambda^\gamma)/u} \right) = \ln\left(\frac{u - u\lambda/q(u)}{u + \lambda - a\lambda^\gamma} \right). \end{aligned}$$

Using L'Hopital's rule, we see that for $a\lambda^\gamma - \lambda > 0$, $\lim_{u\to a\lambda^\gamma-\lambda} \ln\left(\frac{u - u\lambda/q(u)}{u + \lambda - a\lambda^\gamma}\right) < \infty$, since the expression inside the logarithm tends to a finite constant as $u \to a\lambda^\gamma - \lambda$.

Then it is not hard to show that the above equality holds for all $u > 0$ (integrate from u to $a\lambda^\gamma - \lambda - \varepsilon$ and $a\lambda^\gamma - \lambda + \varepsilon$ to infinity and then let $\varepsilon \to 0$). Thus, using (3.1),

$$\int_0^\infty \int_0^\infty e^{-us-\lambda T} d_T(M(s,T))ds = \frac{1-\lambda/q(u)}{u+\lambda-a\lambda^\gamma}.$$

Let $H(s,T) = P\{H(T) \le s\}$ and recall that $M(s,T) = P\{\mathrm{Max}(s) < T\}$. Then $H(s,T) = P\{H(T) \le s\} = P\{\mathrm{Max}(s) \ge T\} = 1 - M(s,t)$, and hence we can integrate by parts to get

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-us-\lambda T} d_T(M(s,T))ds &= \lambda \int_0^\infty \int_0^\infty e^{-us-\lambda T} M(s,T) dT\, ds \\ &= \lambda \int_0^\infty \int_0^\infty e^{-us-\lambda T} (1-H(s,T)) ds\, dT \\ &= 1/u - \lambda \int_0^\infty \int_0^\infty e^{-us-\lambda T} H(s,T) ds\, dT\end{aligned}$$

In other words, the Laplace-Laplace transform of the distribution function of the hitting time process is given by

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-us-\lambda T} H(s,T)\, ds\, dT &= \frac{1}{u\lambda} - \frac{1-\lambda/q(u)}{\lambda(u+\lambda-a\lambda^\gamma)} \\ &= \frac{1-a\lambda^{\gamma-1}+u/q(u)}{u(u+\lambda-a\lambda^\gamma)}. \end{aligned} \tag{3.4}$$

All we have to do is invert the above...

Theorem 3.1. *Let $1 < \gamma \le 2$, let m be the function with Laplace transform $\tilde{m}(u) = 1/q(u)$ given by Lemma 3.1 and g_γ be the maximally skewed γ-stable density;i.e., the Fourier transform of g_γ is $\hat{g}_\gamma(k) = e^{(ik)^\gamma}$. Then*

$$H(s,t) = \int_{\frac{t-s}{(as)^{1/\gamma}}}^\infty g_\gamma(u)\, du + \int_0^s \frac{m(s-u)}{(au)^{1/\gamma}} g_\gamma\left(\frac{t-u}{(au)^{1/\gamma}}\right) du. \tag{3.5}$$

Proof. Clearly, the inverse in u of (3.4) is given by[b]

$$\begin{aligned}\tilde{H}(s,\lambda) &= (1-a\lambda^{\gamma-1}) \int_0^s \exp(-r(\lambda-a\lambda^\gamma))\, dr + \int_0^s m(s-r) \exp(-r(\lambda-a\lambda^\gamma))\, dr \\ &= \frac{1-\exp(-s(\lambda-a\lambda^\gamma))}{\lambda} + \int_0^s m(s-r) \exp(-r(\lambda-a\lambda^\gamma))\, dr.\end{aligned}$$

[b]As the Laplace transform in λ has to stay bounded as $\lambda \to \infty$, we see that convolution with the function m has to have the following effect for all $s > 0$:

$$m \star \exp(-s(\lambda-a\lambda^\gamma)) \approx \exp(-s(\lambda-a\lambda^\gamma))/\lambda.$$

Inverting with respect to λ is a bit more delicate. Using the complex inversion formula (see, for example, Widder[19] Thm 7.6) we obtain

$$\begin{aligned}\int_0^t H(s,r)dr &= \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\Bigg(\frac{1-\exp(-s(\lambda - a\lambda^\gamma))}{\lambda} \\ &\qquad + \int_0^s m(s-r)\exp(-r(\lambda - a\lambda^\gamma))\,dr\Bigg)\,d\lambda \\ &= \frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\left(\frac{1-\exp(-s(\lambda - a\lambda^\gamma))}{\lambda}\right)d\lambda \\ &\quad + \int_0^s m(s-r)\frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\exp(-r(\lambda - a\lambda^\gamma))\,d\lambda\,dr\end{aligned}$$

the second equality holding due to Fubini since

$$\begin{aligned}&\int_0^s |m(s-r)|\int_{c+i\mathbb{R}} \frac{|e^{\lambda t}|}{|\lambda|}|\exp(-r(\lambda - a\lambda^\gamma))|\,d\lambda\,dr \\ &\qquad = e^{ct}\int_0^{s/2}|m(s-r)|\int_{c+i\mathbb{R}}\frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,d\lambda\,dr \\ &\qquad\quad + e^{ct}\int_{s/2}^{s}|m(s-r)|\int_{c+i\mathbb{R}}\frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,d\lambda\,dr \\ &\qquad \le e^{ct}\sup_{s/2<r<s}|m(r)|\int_{c+i\mathbb{R}}\int_0^{s/2}\frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,dr\,d\lambda \\ &\qquad\quad + e^{ct}\|m\|_1\sup_{s/2<r<s}\int_{c+i\mathbb{R}}\frac{|\exp(-r(\lambda - a\lambda^\gamma))|}{|\lambda|}\,d\lambda \\ &\qquad < \infty.\end{aligned}$$

Making a change of variables we see that on the right hand side we have an expression akin to the formula for the inversion of the Fourier transform.

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\left(\frac{1-\exp(-s(\lambda - a\lambda^\gamma))}{\lambda}\right)d\lambda \\ &\quad + \int_0^s m(s-r)\frac{1}{2\pi i}\int_{c+i\mathbb{R}} \frac{e^{\lambda t}}{\lambda}\exp(-r(\lambda - a\lambda^\gamma))\,d\lambda\,dr \\ &= \frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} e^{iut}\left(\frac{1-\exp(-s(iu+c-a(iu+c)^\gamma))}{(iu+c)^2}\right)du \\ &\quad + \int_0^s m(s-r)\frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty} e^{iut}\frac{\exp(-r(iu+c-a(iu+c)^\gamma))}{iu+c}\,du\,dr \qquad (3.6)\end{aligned}$$

Now the Fourier transform of a shifted γ-stable distribution is

$$\int_{-\infty}^{\infty} e^{-ikx}\frac{1}{(ar)^{1/\gamma}}g_\gamma\left(\frac{x-r}{(ar)^{1/\gamma}}\right)dx = e^{-ikr+ra(ik)^\gamma}.$$

Since $e^{-cx}g_\gamma(x)$ is bounded for all $x \in \mathbb{R}$ for some $c \geq 0$ (e.g., see Uchaikin and Zolotarev,[18] Theorem 4.7.1) we obtain that

$$\int_{-\infty}^{\infty} e^{-iux} \frac{e^{-cx}}{(ar)^{1/\gamma}} g_\gamma \left(\frac{x-r}{(ar)^{1/\gamma}} \right) dx = e^{-r(iu+c+a(iu+c)^\gamma)}.$$

Hence, the expressions in (3.6) are indeed inverse Fourier transforms and

$$\begin{aligned} \frac{e^{ct}}{2\pi} & \int_{-\infty}^{\infty} e^{iut} \left(\frac{1-\exp(-s(iu+c-a(iu+c)^\gamma))}{(iu+c)^2} \right) du \\ & + \int_0^s m(s-r) \frac{e^{ct}}{2\pi} \int_{-\infty}^{\infty} e^{iut} \frac{\exp(-r(iu+c-a(iu+c)^\gamma))}{iu+c} \, du \, dr \\ & = t - \int_{-\infty}^{t} \int_{-\infty}^{w} \frac{1}{(as)^{1/\gamma}} g_\gamma \left(\frac{x-s}{(as)^{1/\gamma}} \right) dx \, dw \\ & + \int_0^s m(s-r) \int_{-\infty}^{t} \frac{1}{(ar)^{1/\gamma}} g_\gamma \left(\frac{x-r}{(ar)^{1/\gamma}} \right) dx \, dr. \end{aligned}$$

Therefore,

$$\begin{aligned} H(s,t) &= 1 - \int_{-\infty}^{t} \frac{1}{(as)^{1/\gamma}} g_\gamma \left(\frac{x-s}{(as)^{1/\gamma}} \right) dx + \int_0^s \frac{m(s-r)}{(ar)^{1/\gamma}} g_\gamma \left(\frac{t-r}{(ar)^{1/\gamma}} \right) dr \\ &= \int_{t}^{\infty} \frac{1}{(as)^{1/\gamma}} g_\gamma \left(\frac{x-s}{(as)^{1/\gamma}} \right) dx + \int_0^s \frac{m(s-r)}{(ar)^{1/\gamma}} g_\gamma \left(\frac{t-r}{(ar)^{1/\gamma}} \right) dr \\ &= \int_{\frac{t-s}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u) \, du + \int_0^s \frac{m(s-u)}{(au)^{1/\gamma}} g_\gamma \left(\frac{t-u}{(au)^{1/\gamma}} \right) du. \end{aligned} \tag{3.7}$$

□

Remark 3.1. For $t >> 0$,

$$H(s,t) \approx \int_{\frac{t-s}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u) \, du = P(W(s) \geq t) \tag{3.8}$$

in view of the fact that $W(s)$ is identically distributed with $(as)^{1/\gamma} W_\gamma + s$ where W_γ is the stable random variable with density g_γ. If $W(s)$ were an increasing process, the left-hand and right-hand expressions in (3.8) would be equal. Hence the second term in (3.5) compensates for the fact that $W(s)$ is not monotone.

4. The differential equations

In this section we show that the density of the CTRW limit process solves a fractional partial differential equation with a time derivative of order $1 < \gamma \leq 2$. We begin with a related result concerning the hitting time density. We say that a *mild solution* to a fractional partial differential equation is a function whose Laplace transform solves the equivalent algebraic equation in Laplace-Laplace space. The following theorem employs the Caputo derivative[6,14] $(d/dt)^\gamma$, which can be defined for $1 < \gamma \leq 2$ by requiring that $(d/dt)^\gamma F(t)$ has Laplace transform $\lambda^\gamma \tilde{F}(\lambda) - \lambda^{\gamma-1} F(0) - \lambda^{\gamma-2} F'(0)$ where $\tilde{F}(\lambda)$ is the Laplace transform of $F(t)$.

Theorem 4.1. *There exists a unique distribution f such that the density $u(t,s) = dH(s,t)/ds$ of the hitting time distribution $H(s,t)$ in (3.5) is the unique mild solution to*

$$-a\left(\frac{d}{dt}\right)^{\gamma} u(t,s) + \frac{d}{dt}u(t,s) = -\frac{d}{ds}u(t,s) + f(s)\delta(t) \tag{4.9}$$

with conditions: $u(0,s) = \delta(s); u(t,0) = u_t(0,s) = 0 \ \forall \ s,t > 0$; and $s \mapsto u(t,s)$ is a probability density for all $t > 0$. For any other distribution f equation (4.9) has no solution.

Proof. Assume there exists a solution to (4.9). Using the Caputo derivative and the fact that $u(0,s) = \delta(s)$, the Laplace-Laplace transform of $\left(\frac{d}{dt}\right)^{\gamma} u(t,s)$ is $\lambda^{\gamma}\tilde{\tilde{u}}(\lambda,r) - \lambda^{\gamma-1}$. Then it follows easily that

$$\tilde{\tilde{u}}(\lambda,r) = \frac{1 - a\lambda^{\gamma-1} + \tilde{f}(r)}{r + \lambda - a\lambda^{\gamma}}.$$

Since $s \mapsto u(t,s)$ is a probability density, $|\tilde{u}(t,r)| \le 1$. Hence, the Laplace-Laplace transform for each r is analytic in the right halfplane in λ. Thus $1 - a\lambda^{\gamma-1} + \tilde{f}(r) = 0$ if $r + \lambda - a\lambda^{\gamma} = 0$ or equivalently, $\lambda = q(r)$, q given by Lemma 3.1. Hence

$$\tilde{f}(r) = \frac{-\lambda + a\lambda^{\gamma}}{\lambda} = r/q(r).$$

Since the range of $\lambda \mapsto a\lambda^{\gamma} - \lambda$ contains the right halfplane, $\tilde{f}$ is uniquely determined, and so is f.

The hitting time distribution has Laplace-Laplace transform

$$\int_0^{\infty}\int_0^{\infty} e^{-rs-\lambda T} H(s,T)\,ds\,dT = \frac{1 - a\lambda^{\gamma-1} + r/q(r)}{r(r + \lambda - a\lambda^{\gamma})}$$

by (3.4). Since $H(0,T) = 0$ for all $T > 0$ we have that

$$\int_0^{\infty}\int_0^{\infty} e^{-rs-\lambda T}\, d_s(H(s,T))\,dT = \frac{1 - a\lambda^{\gamma-1} + r/q(r)}{r + \lambda - a\lambda^{\gamma}} \tag{4.12}$$

and hence the density of the hitting-time distribution has the same Laplace-Laplace transform as the solution to the differential equation and it is therefore a mild solution. □

We can now extend this result to include Fourier symbols of semigroup generators. When the CTRW particle jumps are in the generalized domain of attraction of an operator stable limit with probability distribution ν, the long-time limiting particle location process $A(t)$ is operator stable with distribution ν^t.[9] For a simple random walk, the density $p(x,t)$ of $A(t)$ defines a strongly continuous semigroup $(T(t))_{t\ge 0}$ on $L^1(\mathbb{R}^d)$ via the formula $T(t)f(x) = \int f(x-y)p(y,t)dy$. If L is the generator of this semigroup, then $C(x,t) = T(t)f(x)$ also solves the abstract Cauchy problem $dC/dt = LC; C(0) = f$. For a CTRW with infinite mean waiting times, the limiting particle location is given by the subordinated process $A(E(t))$. The density $h(x,t)$ of this process solves a fractional Cauchy problem $(d/dt)^\gamma h = Lh$ where $0 < \gamma < 1$.[3] The next theorem extends this result to $1 < \gamma \leq 2$. Theorem 4.1 in Becker-Kern et al.[5] shows that the long-time CTRW limit process in this case is $A(H(t))$, and Corollary 4.2 in Becker-Kern et al.[5] shows that this limit process has density $\int_0^\infty p(x,s)\, d_s(H(t,s))$. The next theorem shows that this density is the Green's function solution to a fractional partial differential equation (4.13) with time derivative of order $1 < \gamma \leq 2$. This result extends Theorem 3.1 in Baeumer and Meerschaert,[3] showing that these CTRW limits provide stochastic solutions of fractional Cauchy problems with time derivative of order $1 < \gamma \leq 2$.

Theorem 4.2. *Let L be the generator of an operator stable semigroup $(T(t))_{t\geq 0}$ on $L^1(\mathbb{R}^d)$ such that for $f \in L^1(\mathbb{R}^d)$, $Lf = \mathcal{F}^{-1}(\hat{L}(k)f(k))$. Let $a > 0, 1 < \gamma \leq 2$. Then there exists a unique distribution g such that*

$$-a\left(\frac{d}{dt}\right)^\gamma C(t) + \frac{d}{dt}C(t) = LC(t) + \delta(t)g \tag{4.13}$$

with conditions

$$C(0) = f, C'(0) = 0, C(t) \in L^1(\mathbb{R}^d)$$

for all $t \geq 0$ has a mild solution and this unique solution is given by

$$C(t,x) = \int_0^\infty T(s)f(x)\, d_s(H(t,s)).$$

For any other distribution g equation (4.13) has no solution.

Proof. Taking the Fourier transform of (4.13) we obtain

$$-a\left(\frac{d}{dt}\right)^\gamma \hat{C}(t,k) + \frac{d}{dt}\hat{C}(t,k) = \hat{L}(k)\hat{C}(t,k) + \delta(t)\hat{g}(k).$$

Taking the Laplace transform in t yields

$$-a\lambda^\gamma \tilde{\hat{C}}(\lambda,k) + a\lambda^{\gamma-1}\hat{f}(k) + \lambda\tilde{\hat{C}}(\lambda,k) - \hat{f}(k) = \hat{L}(k)\tilde{\hat{C}}(\lambda,k) + \hat{g}(k).$$

Thus

$$\tilde{\hat{C}}(\lambda,k) = \frac{\hat{f}(k) - a\lambda^{\gamma-1}\hat{f}(k) + \hat{g}(k)}{-a\lambda^\gamma + \lambda - \hat{L}(k)}.$$

Since $\tilde{C}(\lambda) \in L^1(\mathbb{R}^d)$ we know that its Fourier transform has to be bounded for all $\Re(\lambda) > 0$. This implies that the numerator of the above equation has to be zero whenever the denominator is equal to zero, or whenever $q(-\hat{L}(k)) = \lambda$. Hence

$$\hat{g}(k) = (-1 + aq(-\hat{L}(k))^{\gamma-1})\hat{f}(k) = -\hat{L}(k)\hat{f}(k)/q(-\hat{L}(k))$$

is uniquely determined. Thus,

$$\tilde{\hat{C}}(\lambda,k) = \frac{1 - a\lambda^{\gamma-1} - \hat{L}(k)/q(-\hat{L}(k))}{-a\lambda^\gamma + \lambda - \hat{L}(k)}\hat{f}(k).$$

To see that $C(t) = \int_0^\infty T(s)f\, d_s(H(t,s))$ has the same Fourier-Laplace transform, take the Fourier transform and observe that

$$\hat{C}(t,k) = \int_0^\infty e^{s\hat{L}(k)}\, d_s(H(t,s))\hat{f}(k).$$

But this is the Laplace transform in s of the hitting time density evaluated at $-\hat{L}(k)$ times $\hat{f}(k)$, i.e. taking the Laplace transform in t, using (4.12) we see that

$$\begin{aligned}\tilde{\hat{C}}(\lambda,k) &= \int_0^\infty \int_0^\infty e^{-\lambda t - s(-\hat{L}(k))}\, d_s(H(t,s))\, dt \hat{f}(k)\\ &= \frac{1 - a\lambda^{\gamma-1} - \hat{L}(k)/q(-\hat{L}(k))}{-a\lambda^\gamma + \lambda - \hat{L}(k)}\hat{f}(k),\end{aligned}$$

and therefore $C(t,x)$ is indeed the mild solution of (4.13).

Finally we prove that g is a distribution. The proof depends on the fact that for some positive real constant C we have

$$|\hat{L}(k)| \le C\max\{\|k\|, \|k\|^2\} \quad \text{for all } k \in \mathbb{R}^d. \tag{4.15}$$

To see this, use the Lévy Representation[10] to write

$$\hat{L}(k) = ik\cdot a - \frac{1}{2}k\cdot Ak + \int_{x\neq 0}\left(e^{ik\cdot x} - 1 - \frac{ik\cdot x}{1+\|x\|^2}\right)\phi(dx)$$

where $a \in \mathbb{R}^d$, A is a nonnegative definite matrix, and ϕ is a σ-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{x\neq 0} \min\{1, \|x\|^2\}\phi(dx) < \infty. \tag{4.16}$$

The integral term I in the Lévy Representation satisfies $|I| \le |I_1| + |I_2|$ with

$$\begin{aligned}I_1 &= \int_{0<\|x\|<1}\left(e^{ik\cdot x} - 1 - \frac{ik\cdot x}{1+\|x\|^2}\right)\phi(dx)\\ &= I_{11} + I_{12}\end{aligned}$$

where

$$\begin{aligned}|I_{11}| &= \left|\int_{0<\|x\|<1}\left(e^{ik\cdot x} - 1 - ik\cdot x\right)\phi(dx)\right|\\ &\le \int_{0<\|x\|<1} \frac{1}{2}\|k\|^2\|x\|^2\phi(dx)\\ &\le C_1\|k\|^2\end{aligned}$$

and

$$\begin{aligned}|I_{12}| &= \left|ik\cdot\int_{0<\|x\|<1}\left(x - \frac{x}{1+\|x\|^2}\right)\phi(dx)\right|\\ &\le \|k\|\int_{0<\|x\|<1}\left(\frac{\|x\|^3}{1+\|x\|^2}\right)\phi(dx)\\ &\le C_2\|k\|\end{aligned}$$

while

$$|I_2| = \int_{\|x\|\geq 1} \left(e^{ik\cdot x} - 1 - \frac{ik \cdot x}{1 + \|x\|^2} \right) \phi(dx)$$
$$\leq D + D + D\|k\|$$

where $D = \phi\{x : \|x\| \geq 1\} < \infty$ using the fact that $|e^{ik\cdot x}| = 1$ and $\|x\|/(1+\|x\|^2) \leq 1$ for $\|x\| \geq 1$. Then (4.15) holds.

Since $\hat{g}(k) = -\hat{L}(k)\hat{f}(k)/q(-\hat{L}(k))$ and $|q(z)| \geq M_0|z|^{1/\gamma}$ for almost all $z \in \mathbb{R}^d$ by (3.3) we have

$$\left| \frac{-\hat{L}(k)}{q(-\hat{L}(k))} \right| \leq \frac{1}{M_0} |\hat{L}(k)|^{1-1/\gamma}$$

and note that $1 - 1/\gamma > 0$. Using (4.15) we obtain

$$|\hat{g}(k)| \leq M_1 |\hat{f}(k)| \max\{\|k\|^{1-1/\gamma}, \|k\|^{2-2/\gamma}\},$$

and then it follows easily that for some $D > 0$ we have

$$\int (1 + \|k\|^2)^{-D} \hat{g}(k) dk < \infty.$$

Now Example 7.12 (b) on p. 191 of Rudin[17] shows that g is a tempered distribution. □

Remark 4.1. Theorem 4.2 extends to any infinitely divisible semigroup $T(t)f(x) = \int f(x-y)\nu^t(dy)$ but in this case the probability distribution ν^t need not have a density.

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