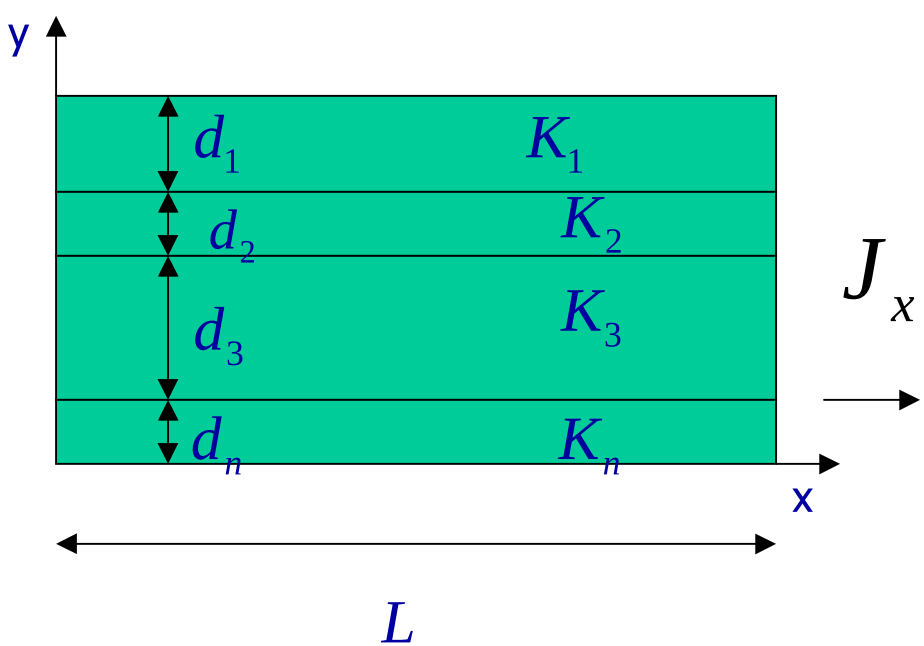
Chapter 3: Permeability of Structured Media

- We can define specific heterogeneous systems and examine the nature of the average conductivity.
- Look at layered media, predict K parallel to layers and perpendicular to layers
- Infer the Permeabilty tensor
- Examine "on/off" permeability grids. Use percolation theory to predict permeability as a function of p (percent "on")
- Darcy's Law is empirical; all ways to derive this law start by assuming a flow field where q is proportion to gradient.
- We may specify a very simple porous medium, e.g. packed spheres, or perhaps bundles of cylinders.
- Using fluid mechanics principles, we can obtain expressions for the frictional resistance and compare them to Darcy's Law.
- The problem with this approach is that it attempts to represent an inherently disordered (random) media with an ordered media
- There may be media that are not well represented by a symmetric permeability tensor

Layered Media

Assume we have a layered media. Each layer has permeability K_i and Thickness d_i.

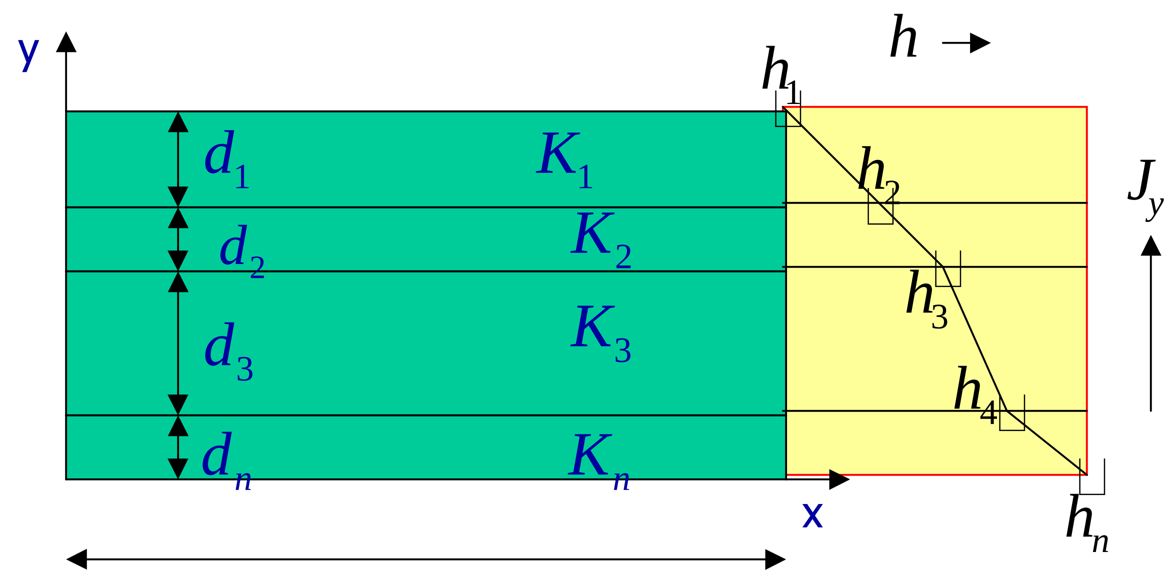


For permeability in the x-direction:

$$Q_x = \sum_{i=1}^{n} K_i d_i J_x = K_{xav} L J_x$$

$$K_{xav} = \frac{1}{L} \sum_{i=1}^{n} K_i d_i$$

For permeability
In the y direction



$$Q_1 = Q_2 = Q_3 = Q_n$$

$$K_1 \frac{h_2 - h_1}{d_1} = K_2 \frac{h_3 - h_2}{d_2} = K_1 \frac{h_3 - h_2}{d_3} = K_n \frac{h_{n+1} - h_n}{d_n}$$

$$Q_{yav} = K_y \frac{h_{n+1} - h_1}{\sum_{i-1,n}^{n} d_i}$$

$$K_{yav} = \frac{Q_y}{\frac{h_{n+1} - h_1}{\sum_{i-1,n}^{n} d_i}}$$

$$K_{yav} = \frac{Q_y}{\dfrac{h_{n+1} - h_1}{\sum_{i=1,n}^{n} d_i}} = \frac{Q_y L}{h_{n+1} - h_1}$$

$$h_{n+1} - h_1 = (h_{n+1} - h_n) + (h_n - h_{n-1}) + \ldots (h_2 - h\ )$$

$$(h_{n+1} - h_n) = Q d_n / K_n$$

$$h_{n+1} - h_1 = Q \sum_{i=1}^{n} \frac{d_n}{K_n}$$

$$K_{yav} = \frac{L}{\sum_{i=1}^{n} \dfrac{d_n}{K_n}}$$

$$K_{xav} = \frac{1}{L}\sum_{i=1}^{n} K_i d_i$$

Conductors in parallel

$$K_{yav} = \frac{L}{\sum_{i=1}^{n} \frac{d_n}{K_n}}$$

Conductors in series

The full permeabilty tensor in principal coordinates is:

$$K_{xy} = \begin{bmatrix} \frac{1}{L}\sum_{i=1}^{n} K_i d & 0 \\ 0 & \frac{L}{\sum_{i=1}^{n} \frac{d_n}{K_n}} \end{bmatrix}$$

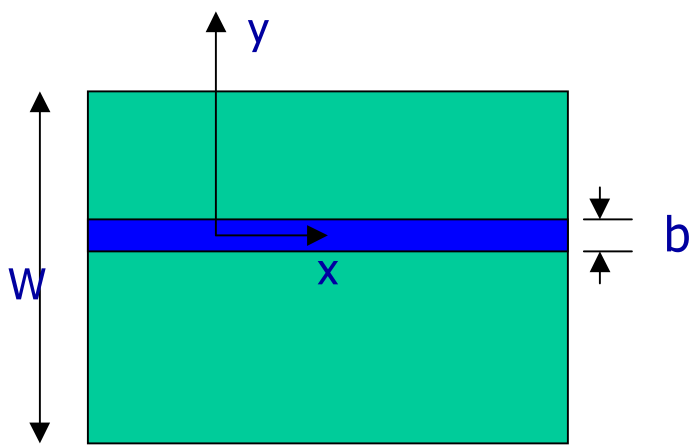
Check your understanding:
What are the x- and y-components of flow for a gradient at θ=30 deg?
Use:

$$K_{xy} = \begin{bmatrix} \frac{1}{L}\sum_{i=1}^{n} K_i d & 0 \\ 0 & \dfrac{L}{\sum_{i=1}^{n} \frac{d_n}{K_n}} \end{bmatrix}$$

and

$$q_i = K_{ij} J_i$$

Permeability in Fracture Networks



Consider a single fracture of aperture, b.

What is the equivalent hydraulic conductivity
Of the block of rock of height,W?

Start with the Navier Stokes Equation
(we cover the derivation later) which describes
Incompressible fluid flow:

$$a_j = \frac{1}{\rho}\frac{\partial p}{\partial x_j} - g\frac{\partial h}{\partial x_j} + \frac{\mu}{\rho}\frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

$a_j =$ acceleration

$\rho =$ density

$p =$ pressure

$g =$ Gravitational const.

$h =$ elevation

$\mu =$ viscosity

$u_i =$ velocity

For creeping flow, acceleration is small compared to viscous terms
Letting

$$\phi = \frac{p}{\rho g} + h$$

$$\frac{\partial \phi}{\partial x_i} = \frac{\mu}{\rho g} \frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

but

$$u_2 = u_3 = 0$$

so

$$\frac{\partial \phi}{\partial x_1} = \frac{\mu}{\rho g} \frac{\partial^2 u_1}{\partial x_3{}^2}$$

Our BC's are for no slip at the fracture surface:

$$u_1(x_3 = b/2) = 0$$

$$u_1(x_3 = -b/2) = 0$$

Double integration over x3 and application of the BC gives the equation for the parabolic Velocity profile in parallel plate flow:

$$u_1 = \rho \frac{g}{\mu} \frac{\partial \phi}{\partial x_1} \left[\frac{x_3{}^2}{2} - \frac{b^2}{8} \right]$$

Integrating u1 over x3 btw –b/2 and b/2:

$$q = -\rho \frac{g}{\mu} \frac{b^3}{12} \frac{\partial \phi}{\partial x_1}$$

The CUBIC LAW

$$\frac{\partial \phi}{\partial x_1} = \frac{\mu}{\rho g} \frac{\partial^2 u_1}{\partial x_3^{\,2}}$$

$$\iint \frac{\partial \phi}{\partial x_1} dx_3 dx_3 = \iint \frac{\mu}{\rho g} \frac{\partial^2 u_1}{\partial x_3^{\,2}} dx_3 dx_3$$

$$\frac{\partial \phi}{\partial x_1} \frac{1}{2} x_3^{\,2} = \frac{\mu}{\rho g} u_1 + C_1 x_3 + C_2$$

Apply the BCs
To solve for C:

$$u_1(x_3 = b/2) = 0$$

$$u_1(x_3 = -b/2) = 0$$

$$C_1 = 0, C_2 = \frac{\partial \phi}{\partial x_1} \frac{1}{2} \frac{b^2}{4}$$

$$\frac{\partial \phi}{\partial x_1} \frac{1}{2} x_3^{\,2} = \frac{\mu}{\rho g} u_1 + \frac{\partial \phi}{\partial x_1} \frac{1}{2} \frac{b^2}{4}$$

$$u_1 = \frac{\rho g}{\mu} \frac{\partial \phi}{\partial x_1} \left[\frac{x_3^{\,2}}{2} - \frac{b^2}{8} \right]$$

$$u_1 = \frac{\rho g}{\mu} \frac{\partial \phi}{\partial x_1} \left[\frac{{x_3}^2}{2} - \frac{b^2}{8} \right]$$

To get the average flow per unit area in the fracture we integrate velocity over the width Of the fracture and divide by the width of the fracture,b

$$\bar{u} = \frac{1}{b} \int_{-b/2}^{b/2} u_1 dx_3 = \frac{1}{b} \int_{-b/2}^{b/2} \frac{\rho g}{\mu} \frac{\partial \phi}{\partial x_1} \left[\frac{{x_3}^2}{2} - \frac{b^2}{8} \right] dx_3$$

$$\bar{u} = -\frac{1}{b} \left[\frac{\rho g}{\mu} \frac{b^3}{12} \frac{\partial \phi}{\partial x_1} \right]$$

$$\bar{u} b = q = -\left[\frac{\rho g}{\mu} \frac{b^3}{12} \frac{\partial \phi}{\partial x_1} \right] = -KA \frac{\partial \phi}{\partial x_1}$$

$$A = b$$

$$K_{fracture} = \frac{\rho g}{\mu} \frac{b^2}{12}$$

So we have the cubic law:

$$q = -\left[\frac{\rho g}{\mu} \frac{b^3}{12} \frac{\partial \phi}{\partial x_1} \right] = -K_{fracture} b \frac{\partial \phi}{\partial x_1}$$

But we want the equivalent permeability for the whole block of rock of width W

$$q = -\left[\frac{\rho g}{\mu} \frac{b^3}{12} \frac{\partial \phi}{\partial x_1} \right] = -K_{block} W \frac{\partial \phi}{\partial x_1}$$

$$K_{block} = \frac{b}{W} \frac{\rho g}{\mu} \frac{b^2}{12} = \frac{b}{W} K_{fracture}$$

$$q = -\frac{\rho g}{\mu}\frac{b^3}{12}\frac{\partial \phi}{\partial x_1}$$

This is the CUBIC LAW

$$K_{fracture} = \frac{\rho g}{\mu}\frac{b^2}{12}$$

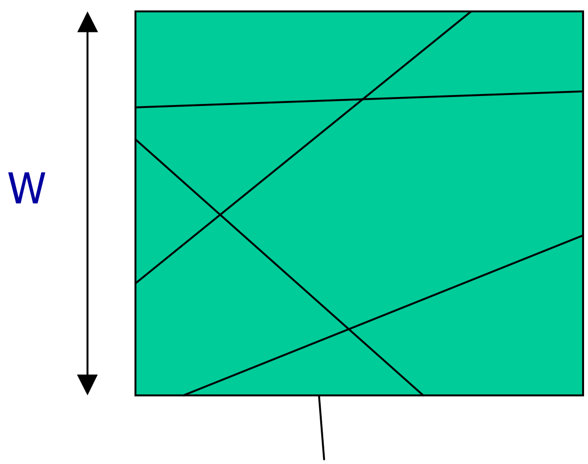
K_f is the equivalent permeability of the fracture

$$q = -K_{fracture} b \frac{\partial \phi}{\partial x_1} = -K_{block} W \frac{\partial \phi}{\partial x_1}$$

$K_{fracture}$ b is sometimes called the equivalent transmissivity of the fracture.

$$K_{fracture} b = K_{block} W$$

Permeability of a set of inifinite, parallel plate fractures (Snow)



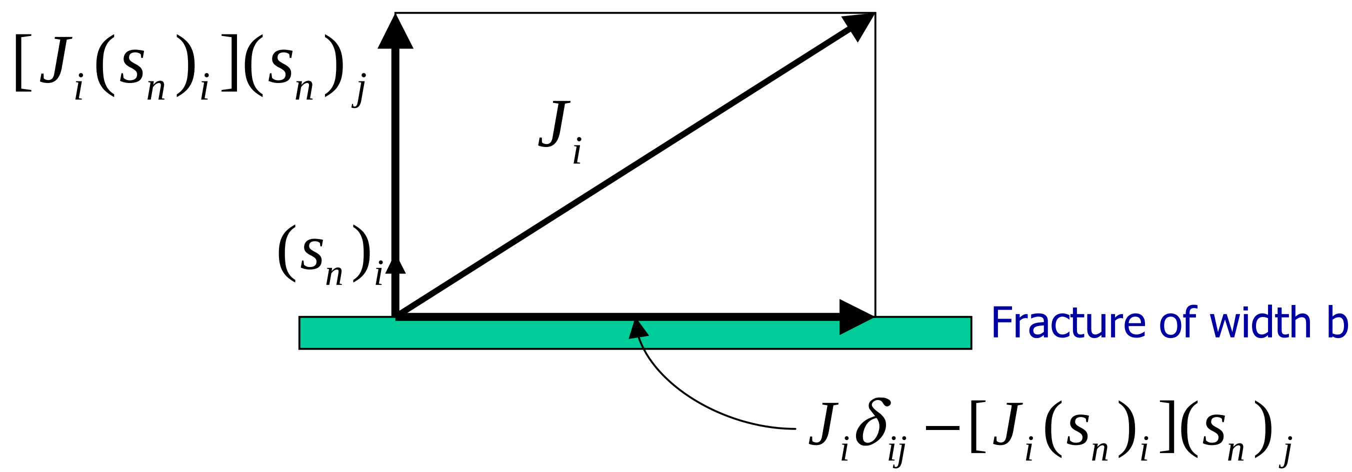
$$K_{ij}W = \frac{1}{12}\frac{\rho g}{\mu}\sum_{n=1}^{N} b_n^{\,3}[\delta_{ij} - (s_n)_i (s_n)_j]$$

$s_n =$ Unit normal to fracture, n

$W =$ Dimension of the rock

$b_n =$ Aperture of fracture n

This equation adds the permeability components of each fracture:
Flow in each fracture depends only on the overall gradient
The network is in "parallel".



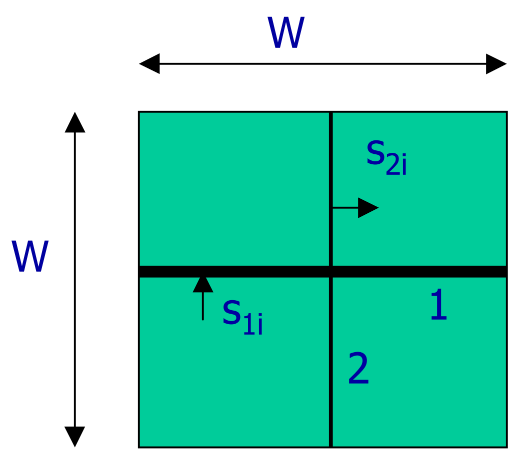
$$J_j = [J_i(s_n)_i](s_n)_j =$$ Component of the gradient normal to the fracture

Using vector subtraction:

$$J_j - [J_i(s_n)_i](s_n)_j = J_i\delta_{ij} - [J_i(s_n)_i](s_n)_j$$

$$= J_i\left[\delta_{ij} - [(s_n)_i](s_n)_j\right] =$$ Component of the gradient in the plane of the fracture

Take an example with two fractures



$$K_{ij}W = \frac{\rho g}{\mu}\frac{1}{12}\sum_{n=1}^{N} b_n^{\,3}(\delta_{ij} - s_{ni}s_{nj})$$

$$W = 100$$

$$b_1 = 10, b_2 = 1$$

$$K_{ij}W = \frac{1}{12}\frac{\rho g}{\mu}\sum_{n=1}^{N} 10^3(\delta_{ij} - s_{ni}s_{nj})$$

$$K_{ij}W = \frac{1}{12}\frac{\rho g}{\mu}\sum_{n=1}^{N} b_n^{\ 3}(\delta_{ij} - s_{ni}s_{nj}); \delta_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; s_{1i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; s_{2i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

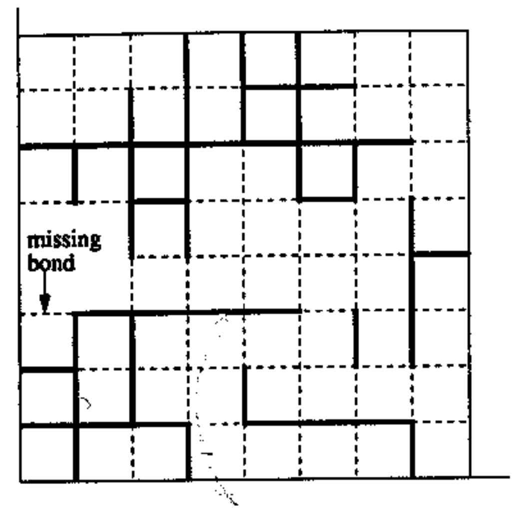
$$100K_{11} = \frac{\rho g}{\mu}\left(\frac{10^3}{12}[1-0] + \frac{1^3}{12}[1-1]\right) = \frac{\rho g}{\mu}\frac{10^3}{12} = K_{n=1}b_1$$

$$100K_{12} = \frac{\rho g}{\mu}\left(\frac{10^3}{12}[0-0] + \frac{1^3}{12}[0-0]\right) = 0$$

$$100K_{22} = \frac{\rho g}{\mu}\left(\frac{10^3}{12}[1-1] + \frac{1^3}{12}[1-0]\right) = \frac{1^3}{12} = K_{n=2}b_2$$

$$K_{ij} = \frac{\rho g}{\mu}\frac{1}{100}\begin{bmatrix} \frac{10^3}{12} & 0 \\ 0 & \frac{1^3}{12} \end{bmatrix} = \frac{1}{W}\begin{bmatrix} K_{n=1}b_1 & 0 \\ 0 & K_{n=2}b_2 \end{bmatrix}$$

Percolation Models: When connectivity becomes an issue



$$\frac{K}{K_{p=1}} \propto (p - p_{crit})^t$$

$p =$ Bond probability

$p_{crit} =$ Critical bond probability

$t =$ Universal exp. ,1<t<1.3

$\propto$ Means "goes as"

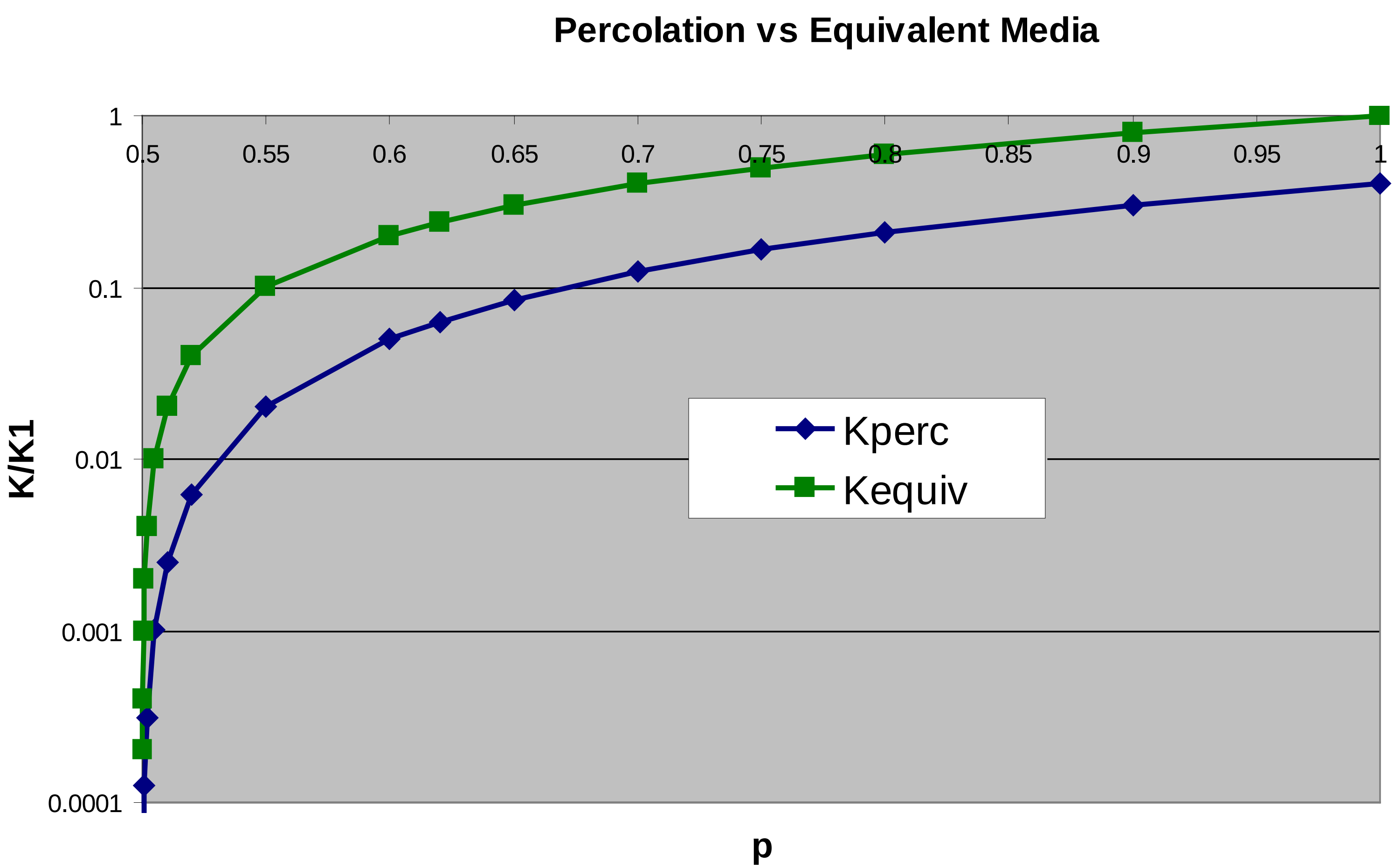
- This equation says that the overall conductance of a partially filled lattice of conductors increases exponentially when $p>p_{crit}$. This equation applies for $p \approx p_{crit}$
- p_{crit} is the level of lattice filling required to get a connected cluster that is infinitely large. Below p_{crit}, there will be finite clusters and K will depend on the scale of measurement.
- Percolation happens suddenly at the critical probability
- This equation has been verified with Monte Carlo studies.

For $p > p_{crit}$ another theory applies: Equivalent Media

$$\frac{K}{K_{p=1}} = 1 - \frac{(1-p)}{(1-2/\zeta)}$$

ζ =The lattice coordination number,
=the number of bonds coordinated with a site
=4 for a square lattice

This says that permeability increases linearly with p if $p > p_{crit}$


Percolation vs Equivalent Media
K/K1
1
0.1
0.01
0.001
0.0001
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
p
Kperc
Kequiv

Percolation Theory and Equivalent Media Theory

- It is possible to develop these equations for a variety of lattices, including random lattices
- Complex lattices, such as with unequal conductances, or randomly oriented elements, can be mapped into simple lattices and the general results apply
- Some think that geologic media are often near the percolation limit
- Percolation theory also looks at the size distribution of connected clusters near the percolation limit and the size of the "backbone" , ie the biggest cluster that will become infinite above the percolation limit
- Percolation theory is useful for two-phase flow. It can predict relative permeability as one phase fills the void and becomes a connected phase where flow can begin

Elementary Flow Tube Models:

- Laminar flow in a tube follows Poiseuille's law:

Flux is linearly proportional to the hydraulic gradient along the tube
The constant of proportionality is a function of

- a geometric factor, G
- density (ρ), viscosity (μ) and g

$$V = G\frac{\rho g}{\mu}\frac{dh}{dx}$$

The approach is

- Assume a pore space geometry,
- Develop an expression for equivalent tubes,
- Add the contributions of tubes of various sizes assuming they carry flow indepently (additively).

Flow Tube Models-- comments

- Flow tube models are often cited to support a “proof” of Darcy’s Law
- Darcy’s law fundamentally says flow is proportional to gradient
- Flow tube models assume that flow is proportional to gradient on the scale of the flow elements and then add the elements
- Flow tube models do no prove Darcy’s Law
- Darcy’s Law is empirical
- Flow tube models are useful for predicting permeability when something is known about the geometry

Pore space in a structured granular media modeled As a bundle of capillary tubes

We start with the pore velocity V:

$$V = \frac{q}{n}$$

Also, recall some basic relationships regarding granular porous medium volumes and areas:

$$V_T = V_s + V_v$$

$$V_v = nV_T$$

$$V_s = (1-n)V_T$$

V_T = total volume

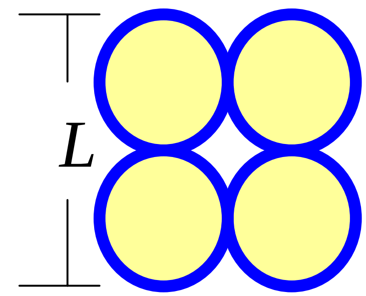
V_s = volume of solids

V_v = volume of voids

The Hydraulic Radius, R_H is defined as the wetted area (A_v) /wetted perimeter (P):

$$R_H = \frac{A_v}{P} = \frac{A_v L}{PL} = \frac{V_v}{A_s}$$

$$A_v L = V_v$$
$$PL = A_s$$



$$V_v = nV_T \qquad V_T = \frac{V_s}{(1-n)} \qquad R_H = \frac{nV_s}{(1-n)A_s}$$

For spherical particles:

$$V_s = \pi d^3$$
$$A_s = 6\pi d^2$$

$$R_H = \frac{nd}{6(1-n)}$$

For aspherical particles:

$$V_s = a\,d^3$$
$$A_s = b\,d^2$$

$$R_H = \frac{n}{(1-n)}\frac{d}{b/a} = \frac{n}{(1-n)}\frac{d}{S}$$

$$R_H = \frac{n}{(1-n)}\frac{d}{S}$$

S is a shape factor ≥ 6

$S = 6$ for perfect spheres

$S \approx 7.7$ for angular particles

Now let's apply this to graded (sorted) materials.

Using a sieve analysis, we can obtain the volume of grains (solids) retained by a particular sieve of size d_i :

$$a n_i d_i^3$$

The total volume is the sum of all size fractions from all the sieves:

$$\sum_{i=1}^{n} a n_i d_i^3$$

The volume fraction, P_i that is held in a particular sieve is:

$$P_i = \frac{a n_i d_i^3}{\sum_{i=1}^{n} a n_i d_i^3}$$

From slide 3, the equation of the hydraulic radius:

$$R_H = \frac{n}{(1-n)} \frac{V_s}{A_s}$$

$$V_s = \sum_{i=1}^{n} a n_i d_i^3 \qquad A_s = \sum_{i=1}^{n} b n_i d_i^2$$

$$R_H = \frac{n}{(1-n)} \frac{\sum_{i=1}^{n} a n_i d_i^3}{\sum_{i=1}^{n} b n_i d_i^2}$$

a and b are constants for all sieve sizes, so they can be factored out

$$R_H = \frac{n}{(1-n)b/a} \frac{\sum_{i=1}^{n} n_i d_i^3}{\sum_{i=1}^{n} n_i d_i^2} \qquad R_H = \frac{n}{(1-n)S} \frac{\sum_{i=1}^{n} n_i d_i^3}{\sum_{i=1}^{n} n_i d_i^2}$$

Take reciprical:

$$R_H^{-1} = \frac{(1-n)S}{n} \frac{\sum_{i=1}^{n} n_i d_i^2}{\sum_{i=1}^{n} n_i d_i^3}$$

$$R_H^{-1} = \frac{(1-n)S}{n} \left[\frac{n_1 d_1^2}{\sum_{i=1}^{n} n_i d_i^3} + \frac{n_2 d_2^2}{\sum_{i=1}^{n} n_i d_i^3} + \ldots + \frac{n_N d_N^2}{\sum_{i=1}^{n} n_i d_i^3} \right]$$

Now multiply and divide by d_i :

$$R_H^{-1} = \frac{(1-n)S}{n}\frac{d_i}{d_i}\left[\frac{n_1 d_1^2}{\sum_{i=1}^{n} n_i d_i^3} + \frac{n_2 d_2^2}{\sum_{i=1}^{n} n_i d_i^3} + \ldots + \frac{n_N d_N^2}{\sum_{i=1}^{n} n_i d_i^3}\right]$$

Distribute the d_i in the numerator into the parentheses:

$$R_H^{-1} = \frac{(1-n)S}{n}\frac{}{d_i}\left[\underbrace{\frac{n_1 d_1^3}{\sum_{i=1}^{n} n_i d_i^3}}_{P_1} + \underbrace{\frac{n_2 d_2^3}{\sum_{i=1}^{n} n_i d_i^3}}_{P_2} + \ldots + \underbrace{\frac{n_N d_N^3}{\sum_{i=1}^{n} n_i d_i^3}}_{P_N}\right] \Rightarrow \sum_{i=1}^{N} P_i$$

and recall :

$$P_i = \frac{n_i d_i^3}{\sum_{i=1}^{n} n_i d_i^3}$$

$$R_H^{-1} = \frac{(1-n)S}{n} \sum_{i=1}^{N} \frac{P_i}{d_i}$$

or :

$$R_H = \frac{n}{(1-n)} \frac{1}{S \sum_{i=1}^{N} \frac{P_i}{d_i}}$$

Head loss in a pipe (from your fluid mechanics course) :

$$h_L = f \frac{L}{D} \frac{V^2}{2g}$$

f is the friction factor

$D = 4 R_H$ for laminar flow; R_H is the hydraulic radius

V is the velocity

g is the acceleration of gravity

Divide through by L :

$$\frac{h_L}{L} = f\,\frac{V^2}{D2g}$$

And let :

$$\frac{h_L}{L} = \frac{dh}{dx} = f\,\frac{V^2}{D2g}$$

$$\left.\begin{array}{l} f = \dfrac{64\mu}{N_R} = \dfrac{64\mu}{VD\rho} \qquad \left(N_R = \dfrac{\rho VD}{\mu} \right) \end{array}\right\}$$

for laminar flow

$$\frac{dh}{dx} = \frac{64\mu}{VD\rho}\,\frac{V^2}{D2g} = \frac{32\mu V^2}{D^2\rho g}$$

$$\frac{dh}{dx} = \frac{64\mu}{VD\rho}\frac{V^2}{D2g} = \frac{32\mu V}{D^2\rho g}$$

Recalling that

$$D = 4R_H$$

$$\frac{dh}{dx} = \frac{2\mu V}{R_H^2 \rho g}$$

and solving for V :

$$V = \frac{R_H^2}{2}\frac{\rho g}{\mu}\frac{dh}{dx}$$

If we apply this to a bundle of capillary tubes (a type of porous media), the 2 becomes a constant, which we will call m :

$$V = \frac{R_H^2}{m}\frac{\rho g}{\mu}\frac{dh}{dx}$$

Does this look familiar?

$$V = \frac{R_H^2}{m}\frac{\rho g}{\mu}\frac{dh}{dx}$$

Darcy's Law:

$$V = \frac{k}{n}\frac{\rho g}{\mu}\frac{dh}{dx}$$

Therefore, we can see that the intrinsic permeability, k, is analogous to R_H :

$$k = \frac{1}{m}\left[\frac{n}{(1-n)}\frac{1}{S\sum_{i=1}^{N}\frac{P_i}{d_i}}\right]^2$$

$$V = \frac{k}{n} \frac{\rho g}{\mu} \frac{dh}{dx}$$

$$V = \frac{K}{n} \frac{dh}{dx}$$

$$K = \frac{\rho g}{\mu} \left[\frac{n}{(1-n)} \frac{1}{S \sum_{i=1}^{N} \frac{P_i}{d_i}} \right]^2$$

or

$$K = \frac{\rho g}{\mu} \frac{n^2}{(1-n)^2} \frac{1}{\left[S \sum_{i=1}^{N} \frac{P_i}{d_i} \right]^2}$$